

THE HIDDEN GEOMETRY OF
RANDOM STEPPED SURFACES

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Minimize area: $h : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\min_h \iint_U \sqrt{1 + h_x^2 + h_y^2} dx dy$$

Here $\sigma(s, t) = \sqrt{1 + s^2 + t^2}$ is the “surface tension”.

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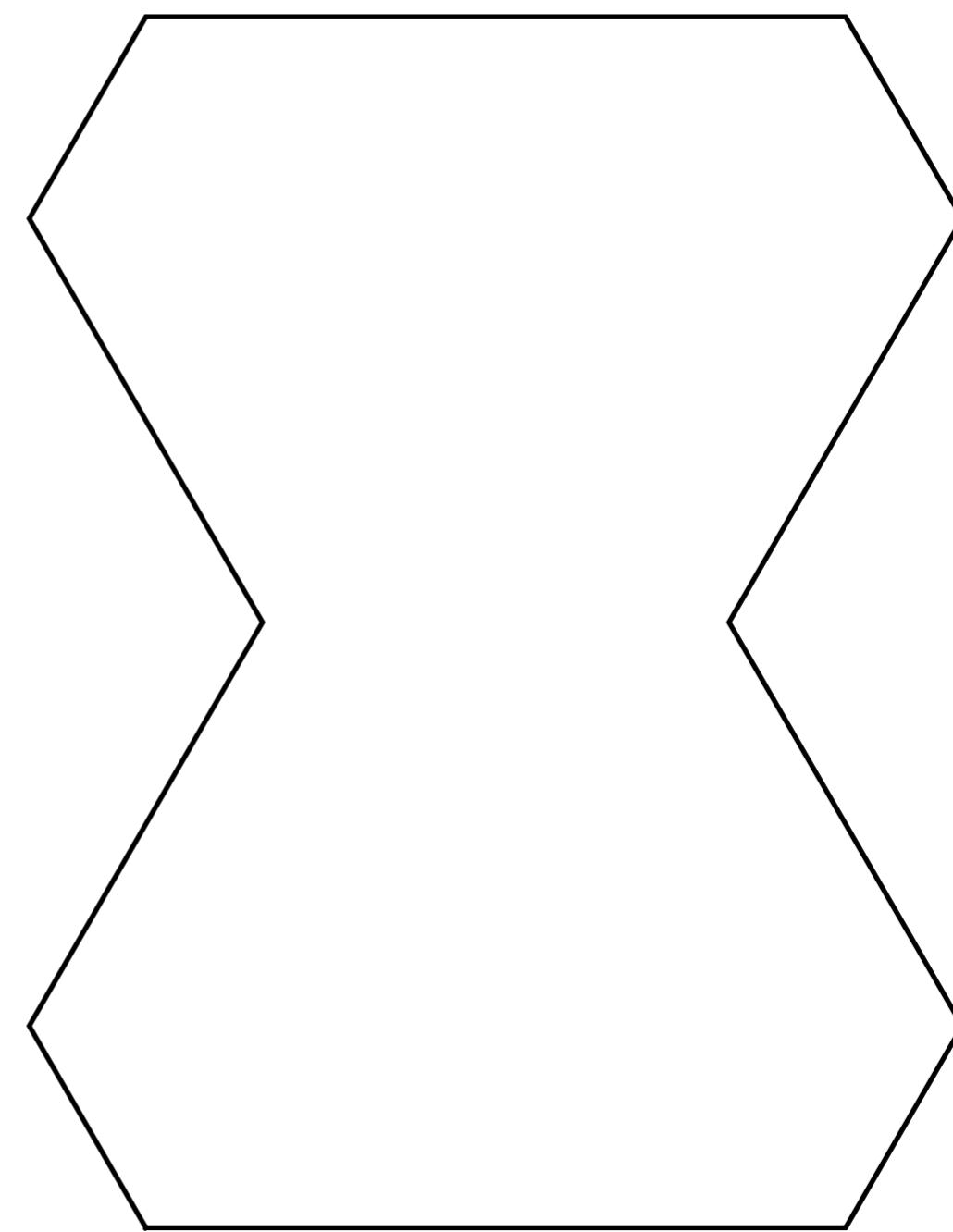
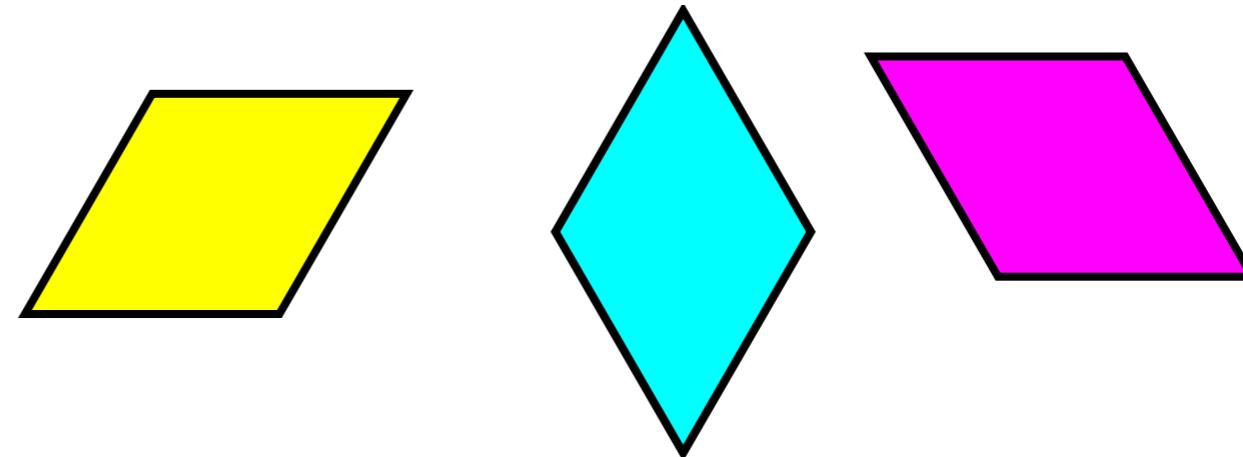
Weierstrass-Enneper parameterization of minimal surfaces

Let f, g be (arbitrary) analytic functions, then

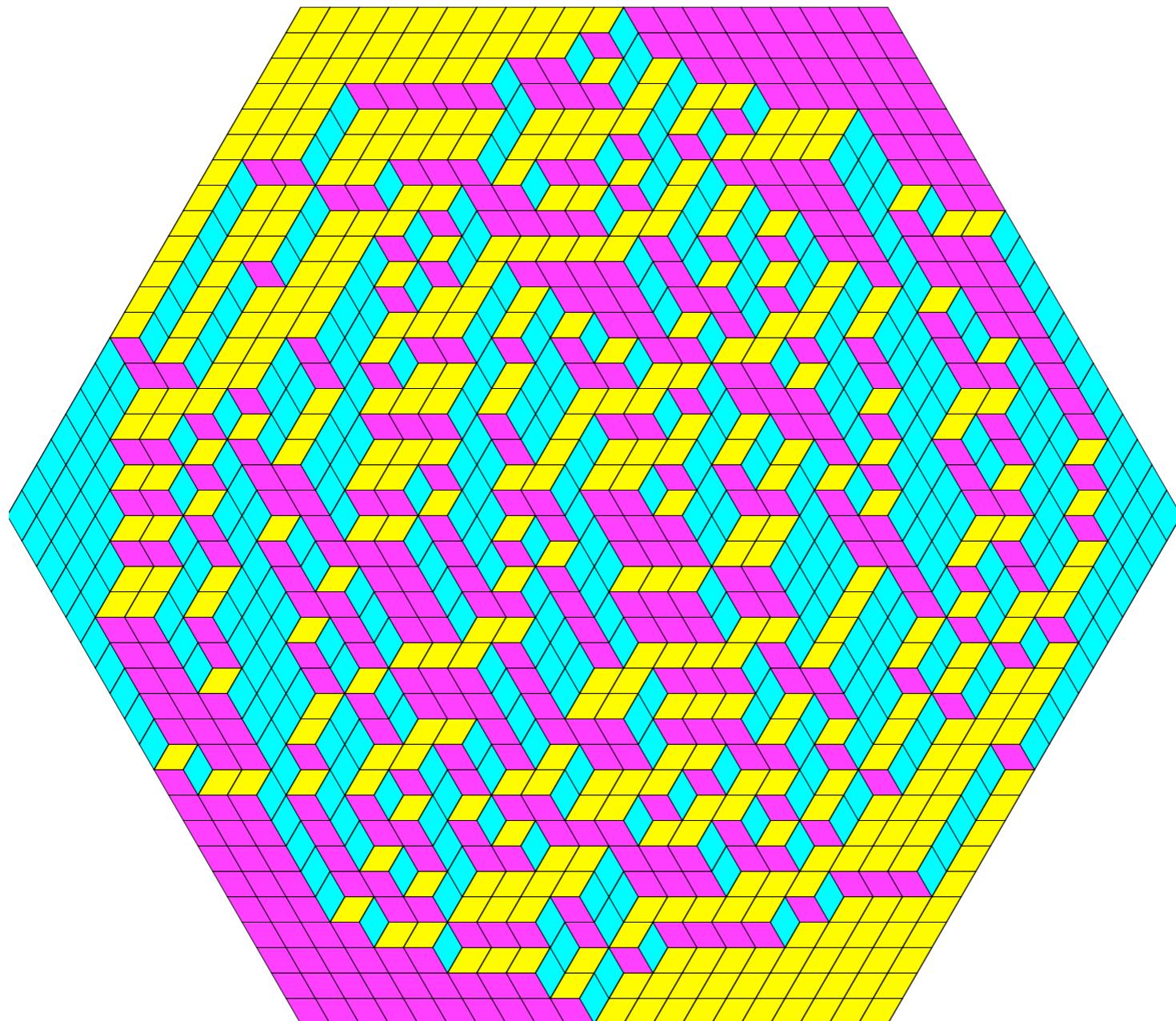
$$Re \left(\int f(z)(1 - g(z)^2) dz, i \int f(z)(1 + g(z)^2) dz, \int f(z)g(z) dz \right)$$

parameterizes a minimal surface in \mathbb{R}^3 .

“Lozenge” tilings



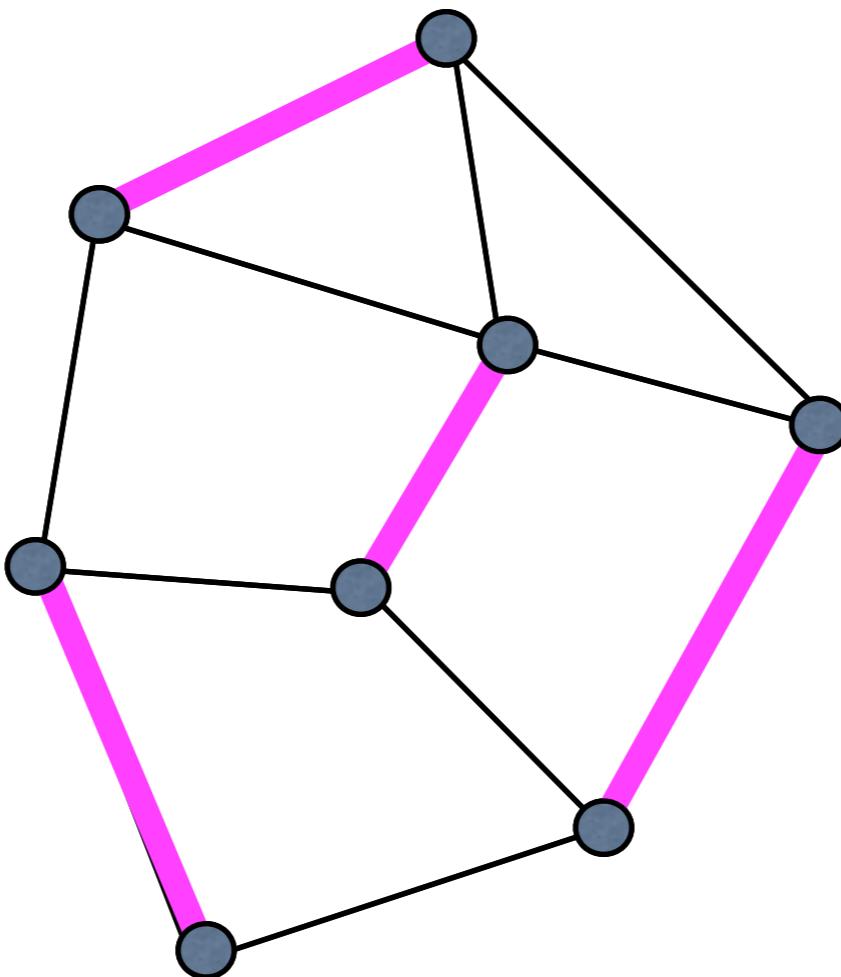
lozenge tilings are stepped surfaces



What happens for large system size?

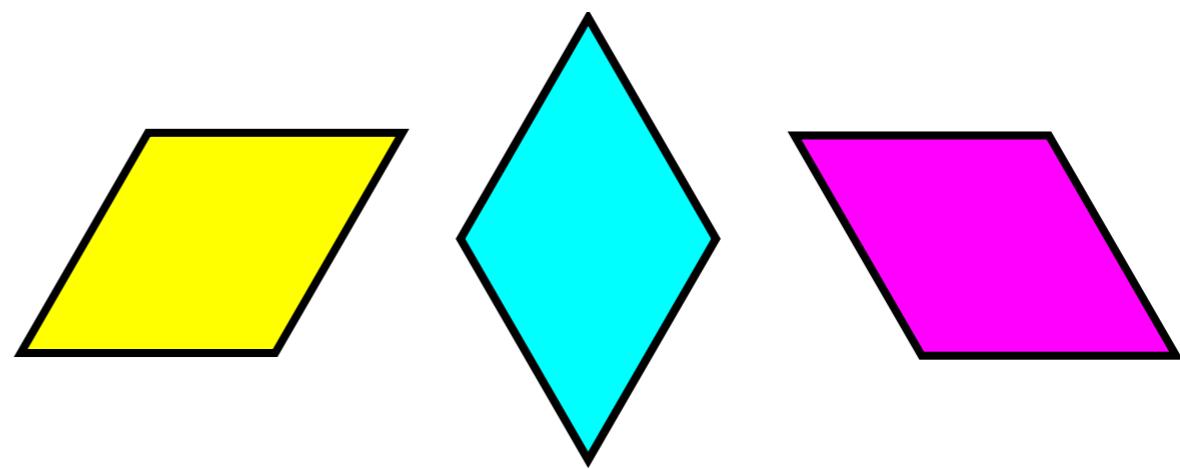
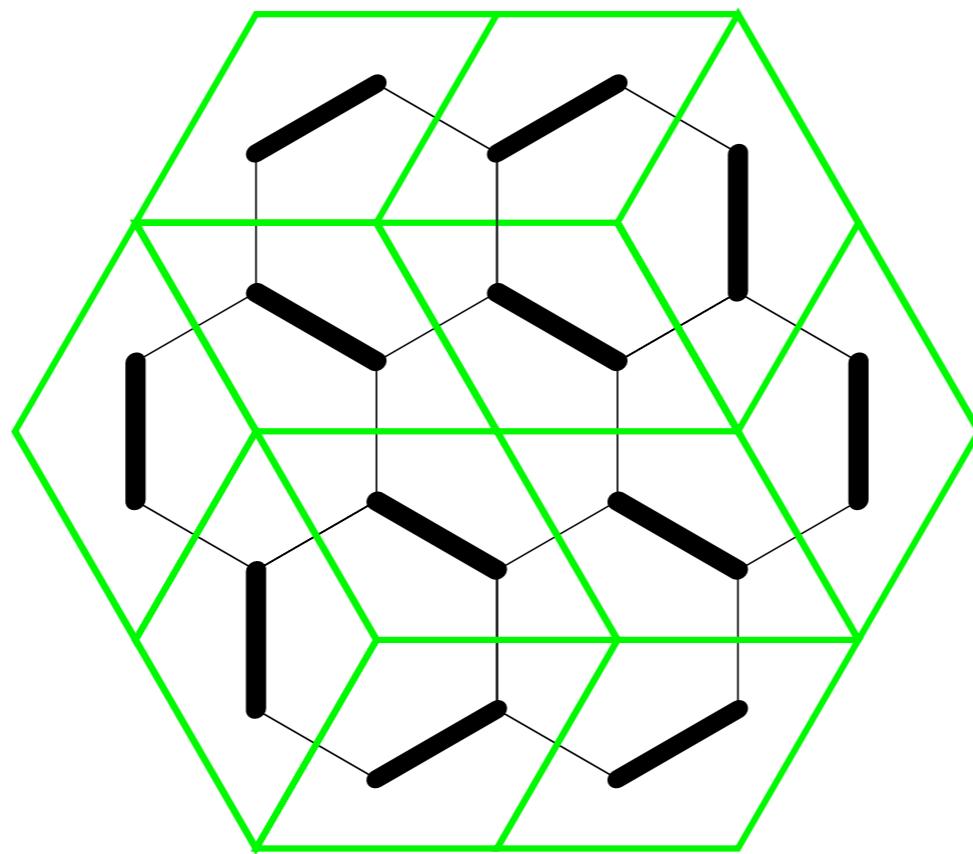
The Dimer Model

Dimer cover of a graph:



perfect matching of the vertices.

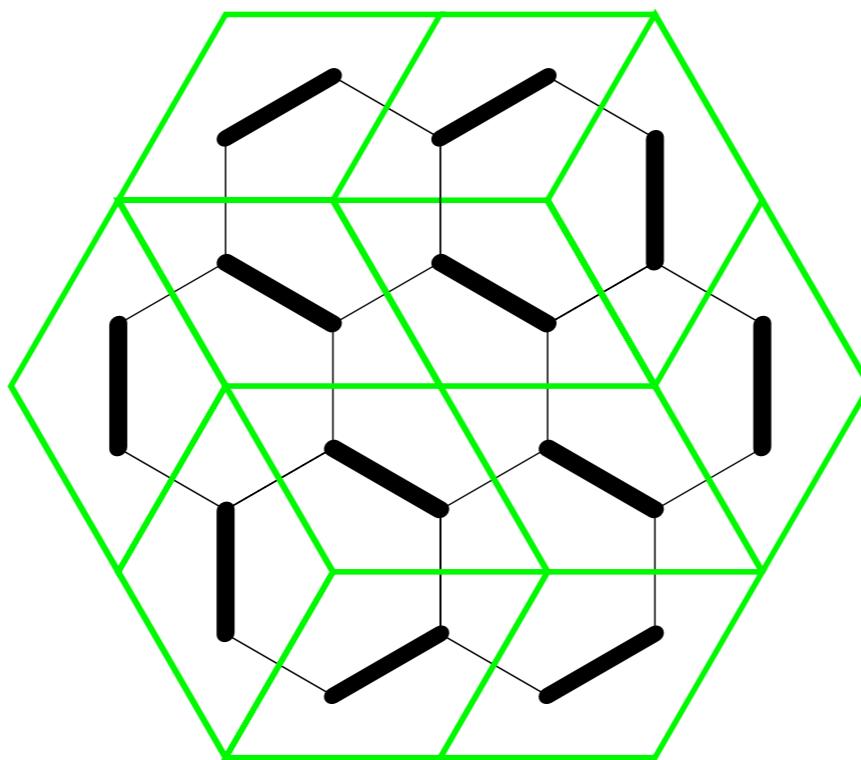
Dimers on honeycomb lattice



Thm[Kasteleyn 1965] For G a part of the honeycomb graph, let K be the bipartite adjacency matrix,

$$K_{wb} = \begin{cases} 1 & b \sim w \\ 0 & \text{else.} \end{cases}$$

Then $|\det K|$ is the number of dimer covers.

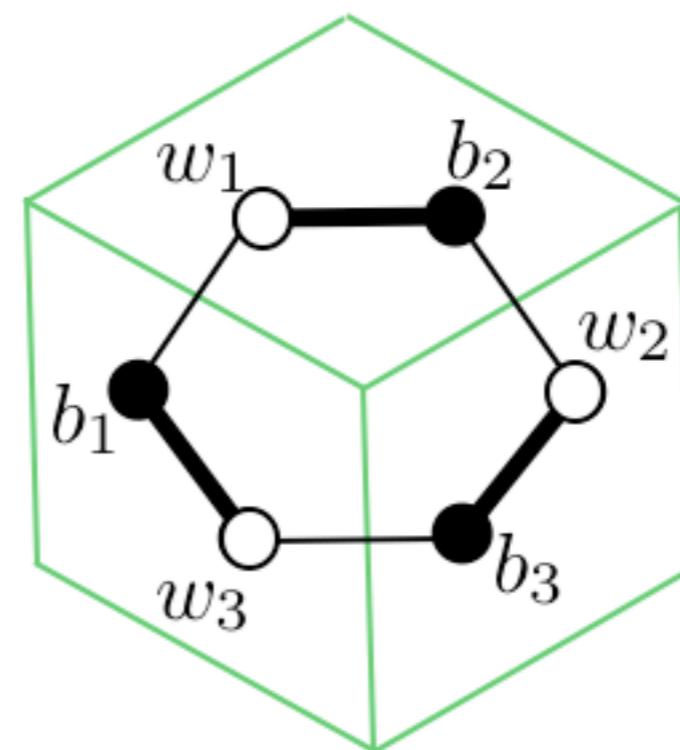
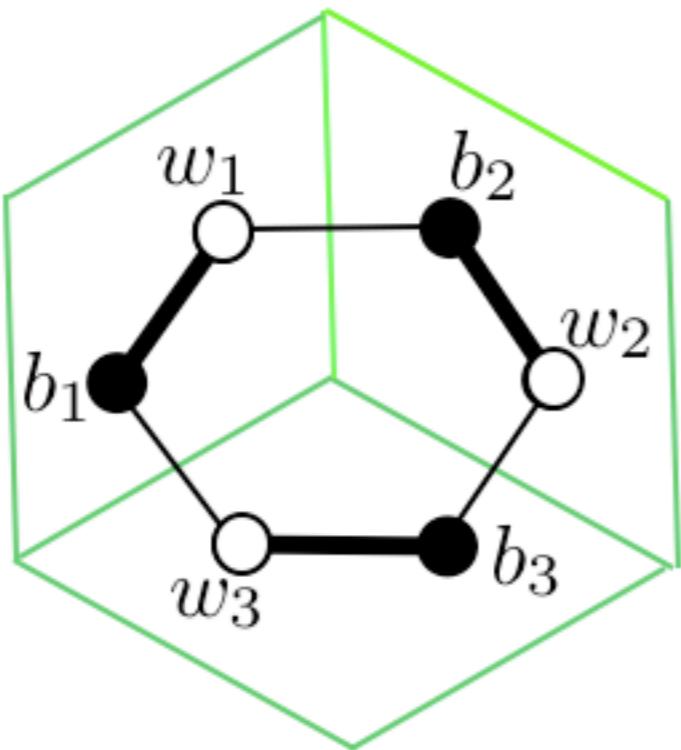


Example: K is 12×12 ; $\det K = 20$.

Proof:

$$\det K = \sum_{\sigma \in S_n} (-1)^\sigma k_{w_1 b_{\sigma(1)}} k_{w_2 b_{\sigma(2)}} \dots k_{w_n b_{\sigma_n}}$$

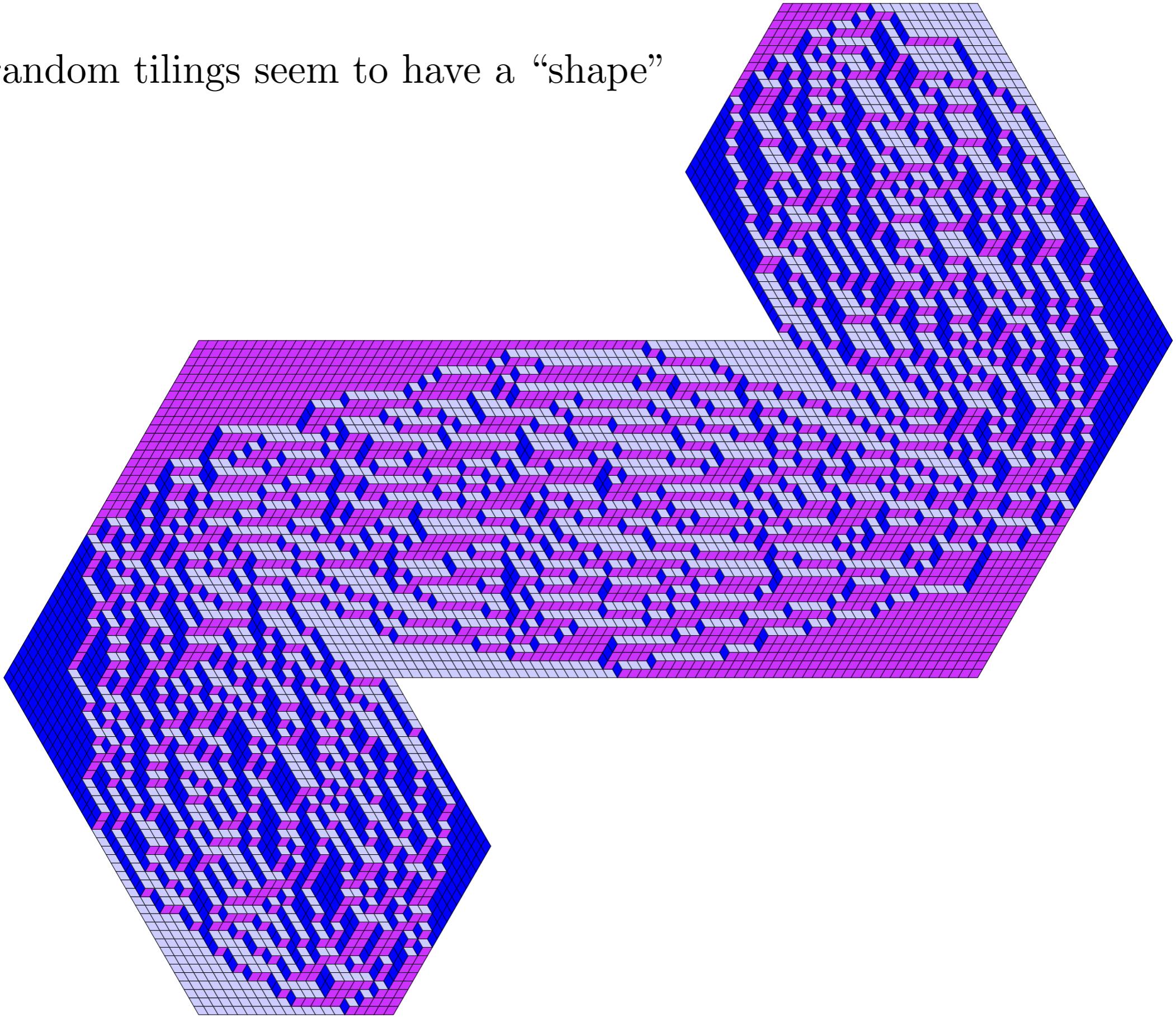
This sum has one nonzero term (± 1) for each dimer covering.

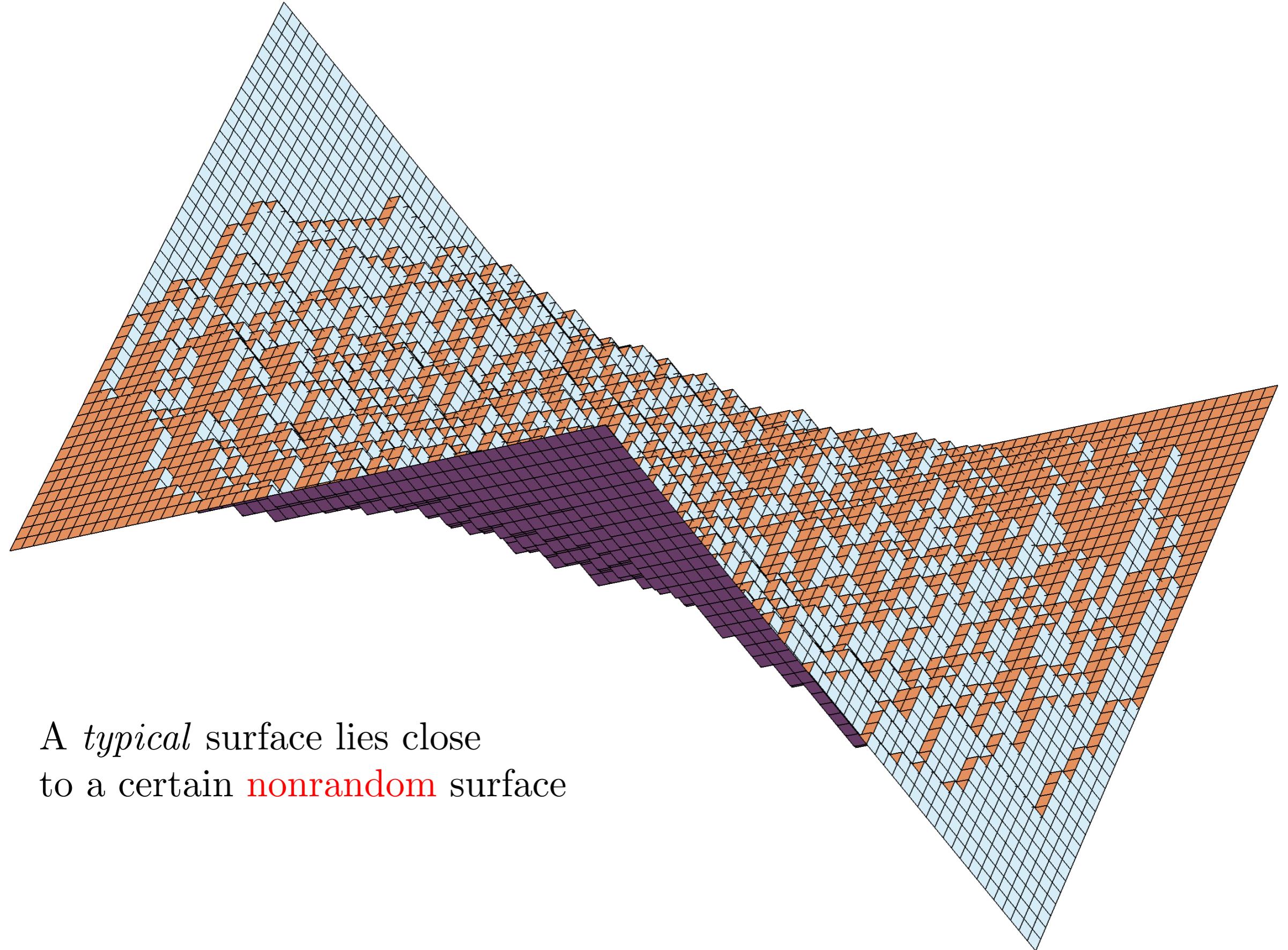


$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma(123))$$

We can get from one dimer cover to any other using these local flips. \square

Large random tilings seem to have a “shape”



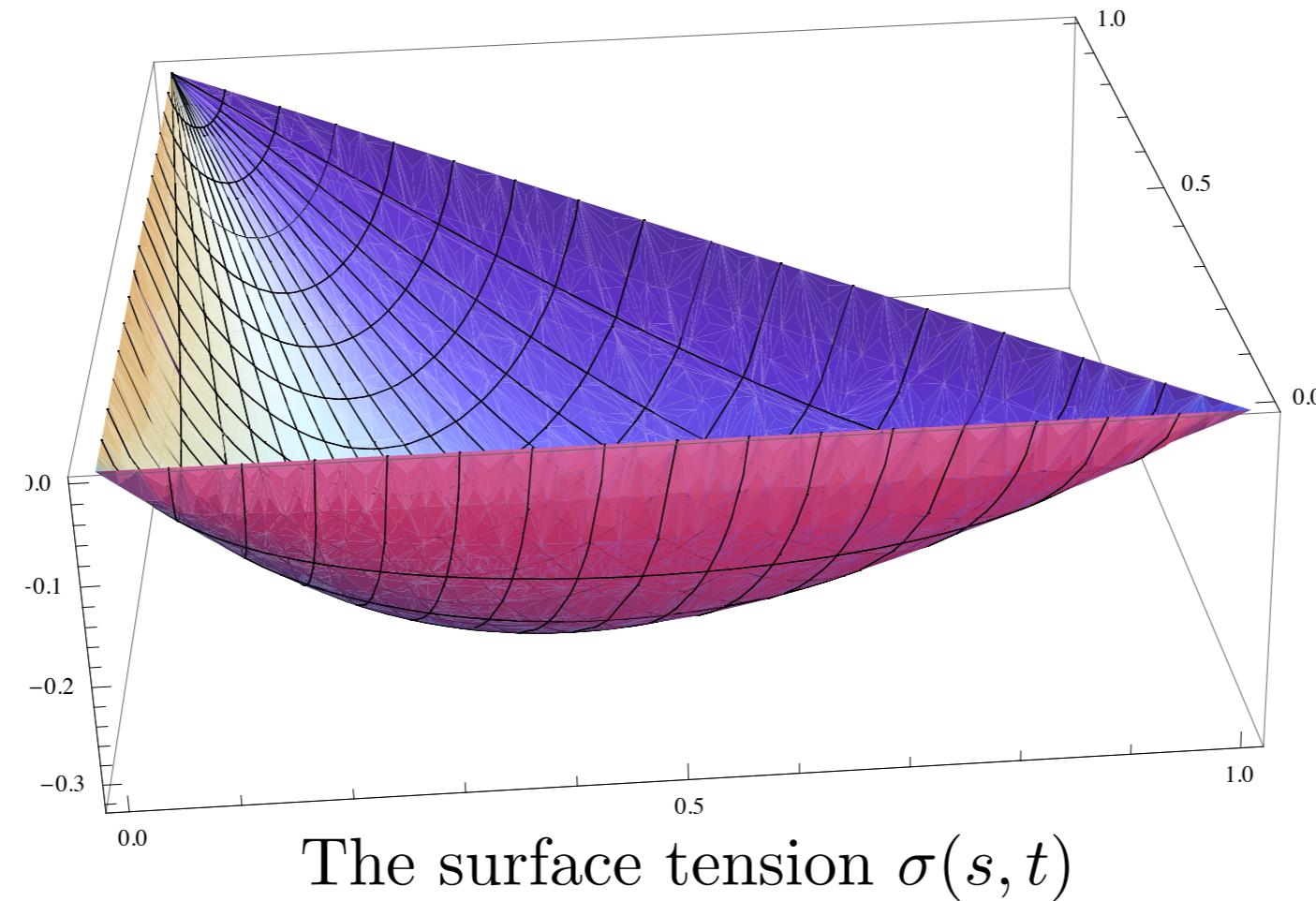


A *typical* surface lies close
to a certain **nonrandom** surface

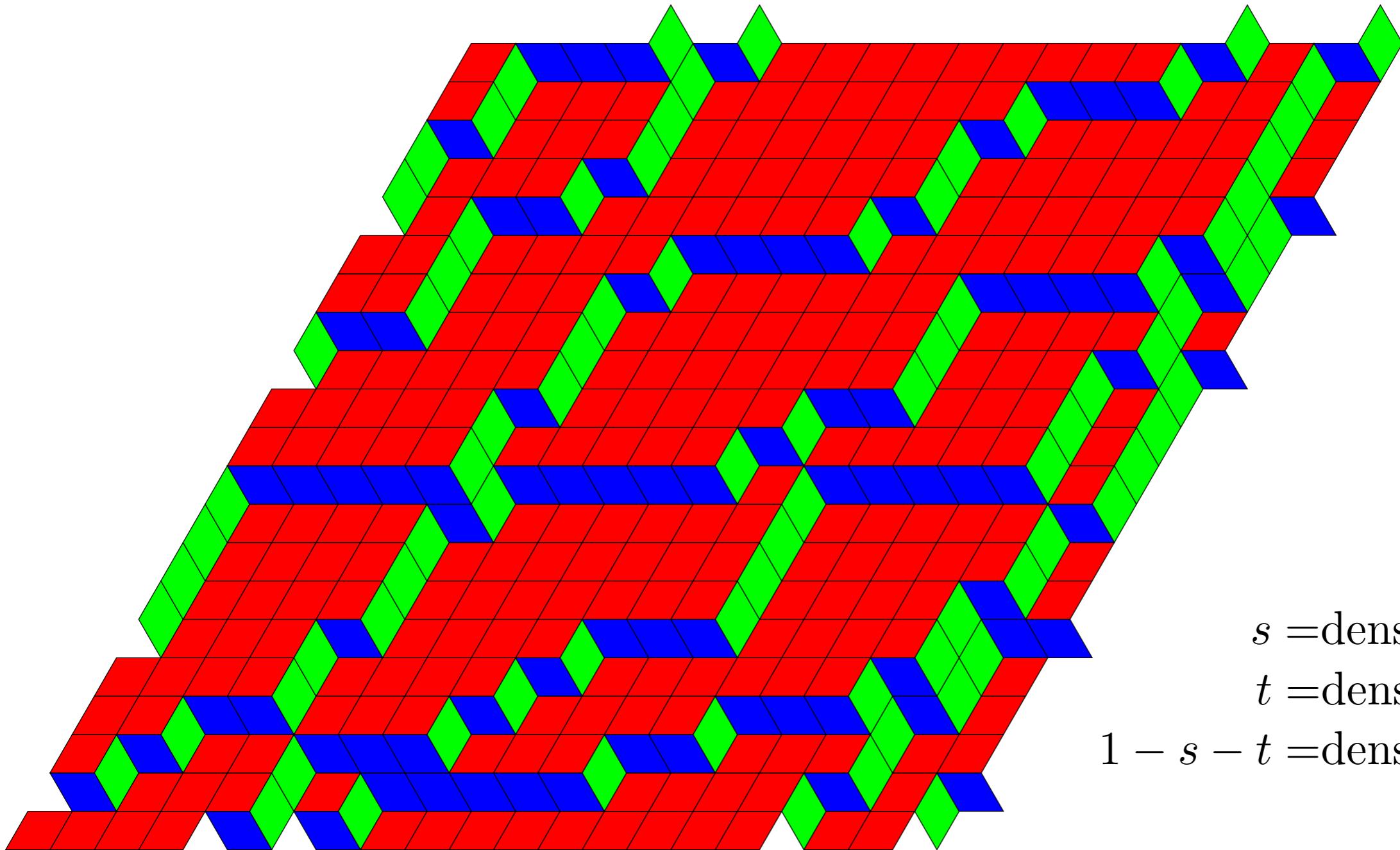
Lozenge tiling limit shape

Thm[Cohn,K,Propp (2000)] The function $h : R \rightarrow \mathbb{R}$ describing the limit shape is the unique minimizer of the surface tension integral

$$\min_h \iint_R \sigma(h_x, h_y) dx dy.$$



$$\sigma(s, t) = \text{vol}_{\mathbb{H}^3}(\text{ideal tetrahedron with dihedral angles } (\pi s, \pi t, \pi(1 - s - t)))$$



for each slope (s, t) there is an associated growth rate (entropy) $-\sigma(s, t)$:

$$(\text{Number of tilings}) = e^{-\text{Area} \cdot \sigma(s, t)(1+o(1))}$$

Alternatively, give tiles weights a , b , c depending on orientation.

Then choose a tiling with probability proportional to its weight $a^{N_a} b^{N_b} c^{N_c}$.

How is σ calculated?

First compute the “free energy” F (the normalized log-of-determinant).

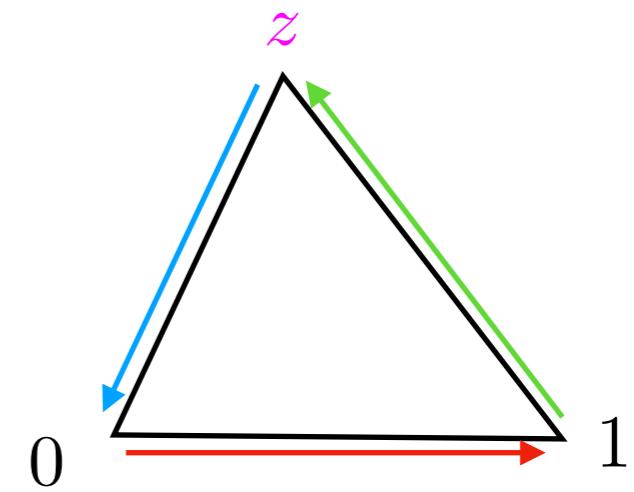
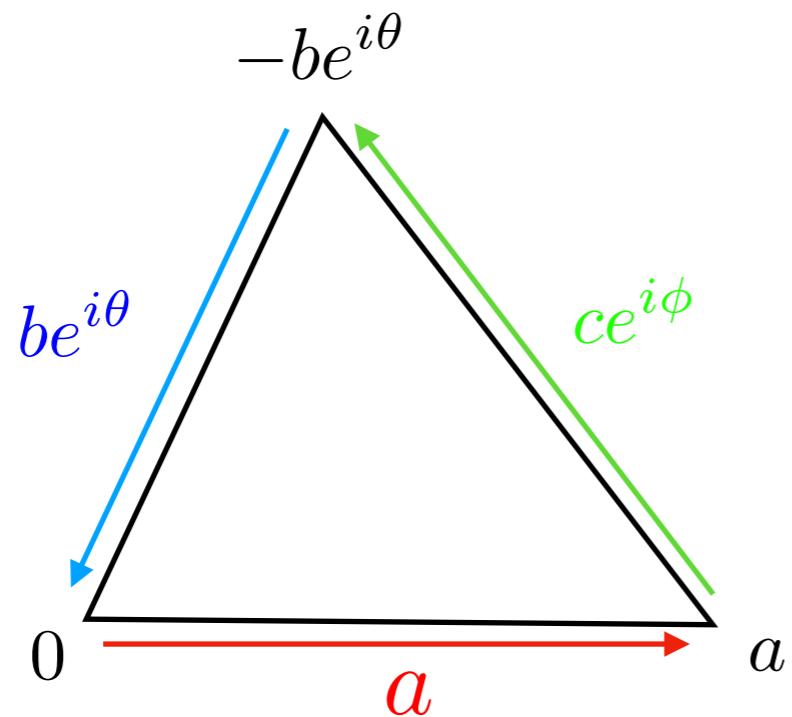
$$F(a, b, c) = \frac{1}{4\pi^2} \iint_{[0, 2\pi]^2} \log(a + \underbrace{be^{i\theta} + ce^{i\phi}}_{\text{eigenvalues of } K}) d\theta d\phi$$

$$\sigma(s, t) = -F(a, b, c) + (1 - s - t) \log(a) + s \log(b) + t \log(c)$$

is the Legendre dual of F

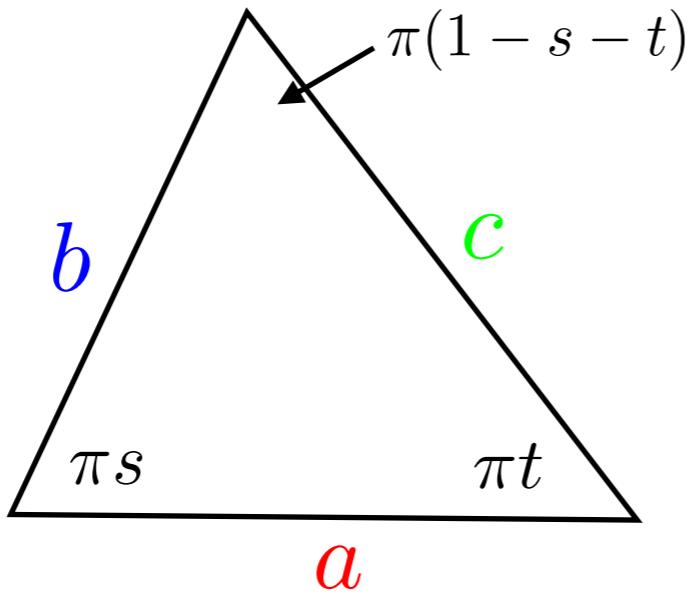
$$F(a, b, c) = \frac{1}{4\pi^2} \iint_{[0, 2\pi]^2} \log(\underbrace{a + be^{i\theta} + ce^{i\phi}}_{\text{eigenvalues of } K}) d\theta d\phi$$

There is a special point (θ, ϕ) where $a + be^{i\theta} + ce^{i\phi} = 0$.



$$z = -\frac{b}{a} e^{i\theta}$$

How are weights related to densities?



s = density of green
 t = density of blue
 $1 - s - t$ = density of red

In particular,

If $a \geq b + c$, system is “frozen” in all red.

if $b \geq a + c$, system is frozen in all blue;

if $c \geq a + b$, system is frozen in all green.

How to solve the variational problem?

The Euler-Lagrange equation for a gradient model is

$$\operatorname{div}_{x,y}(\nabla_{s,t}\sigma(\nabla_{x,y}h)) = 0$$

...a nonlinear PDE.

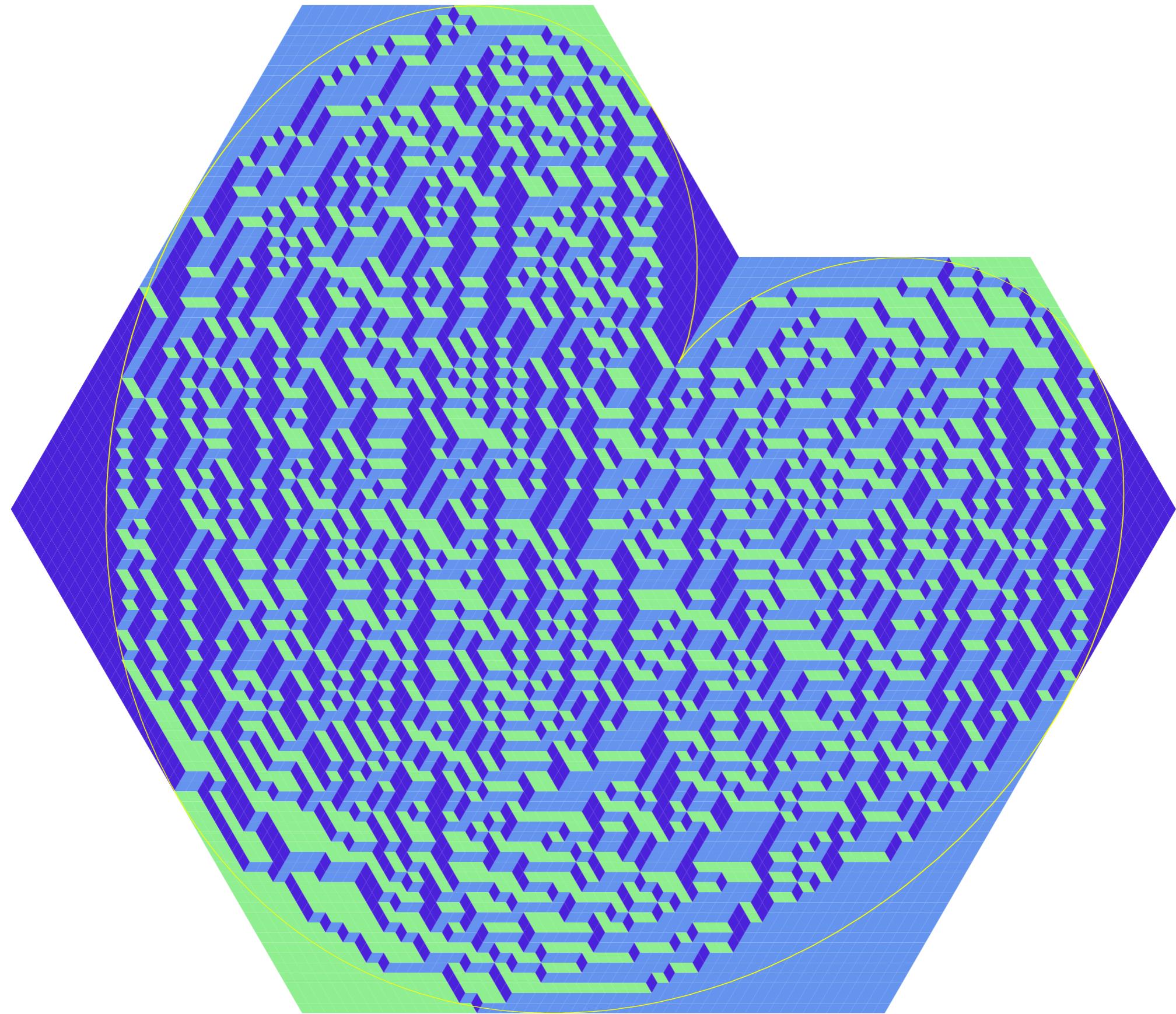
Idea (Ampère): rewrite in terms of the **conformal coordinate** z (or w).

...

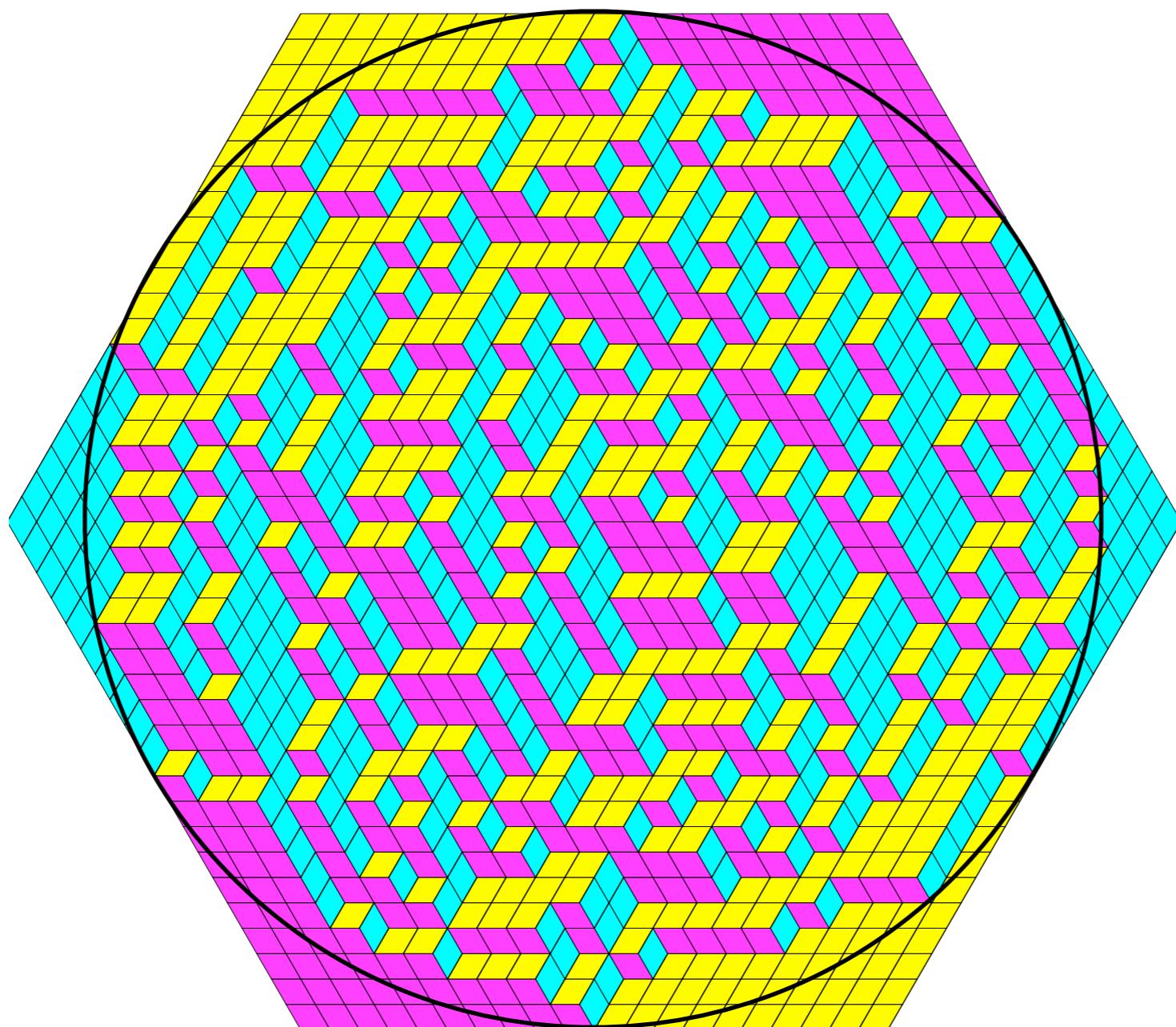
Thm[K-Okounkov] $z = z(x, y)$ is a minimizer iff

$$y = \frac{z}{z-1}x + f(z)$$

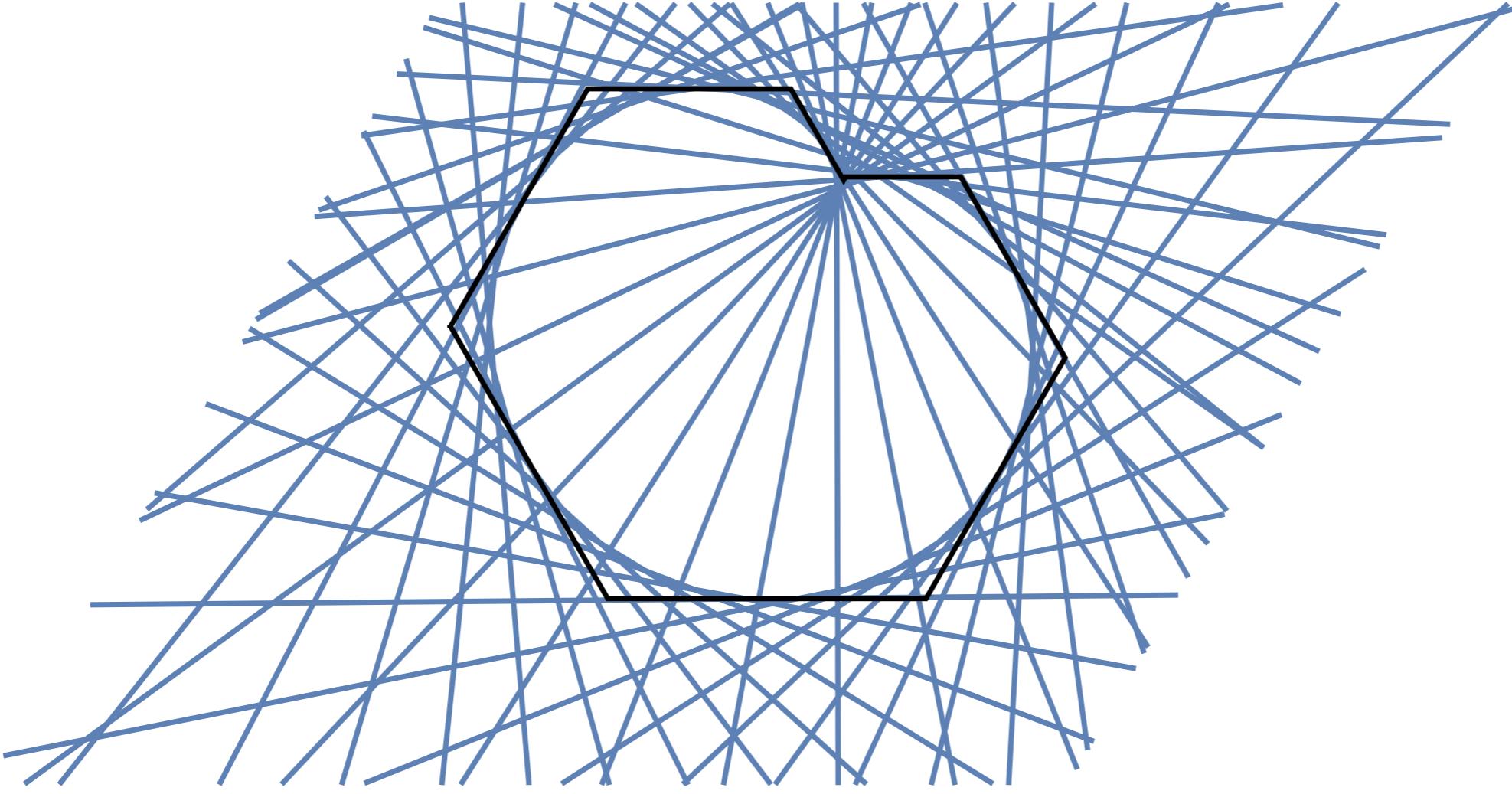
for some analytic function f .



One can solve the minimization problem for “polygonal” boundary conditions...
in which case the arctic boundary is an algebraic curve.



Arctic circle theorem [Jockusch-Propp-Shor, Cohn-Larsen-Propp]



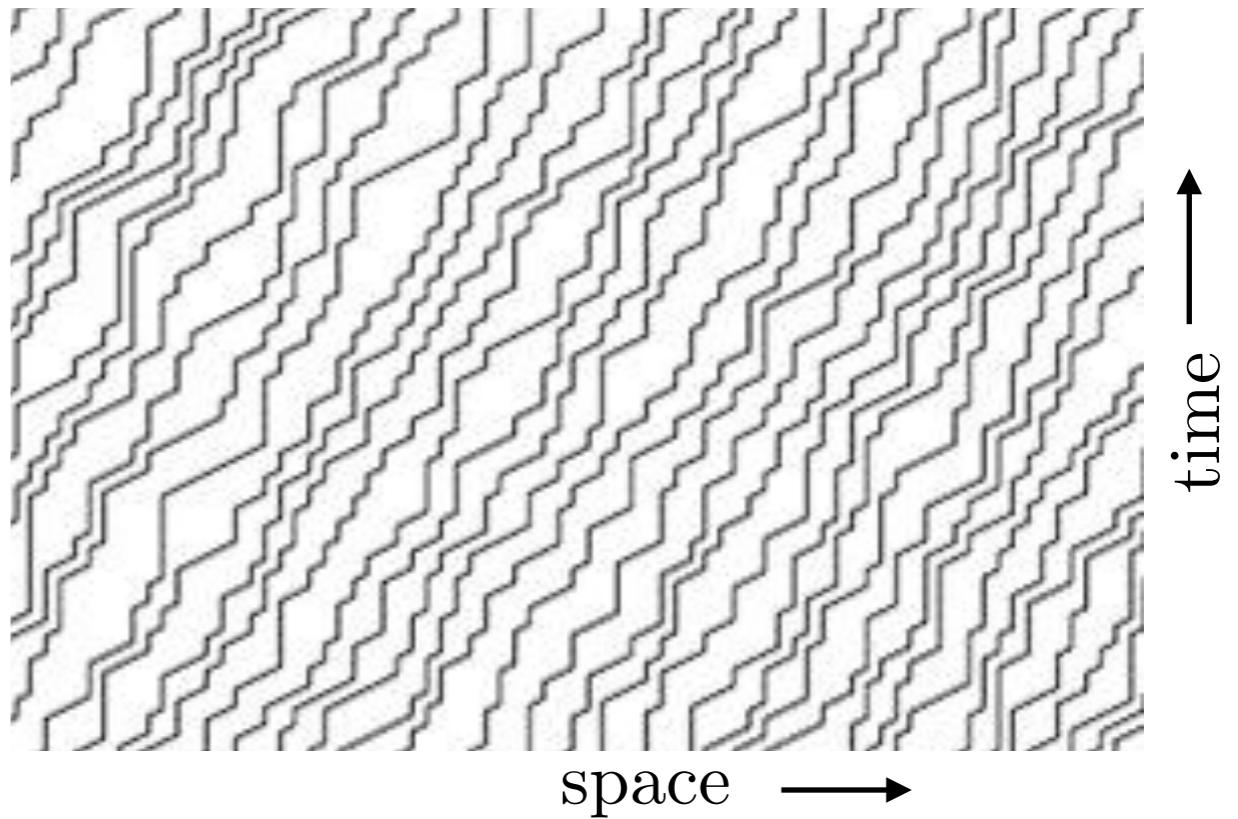
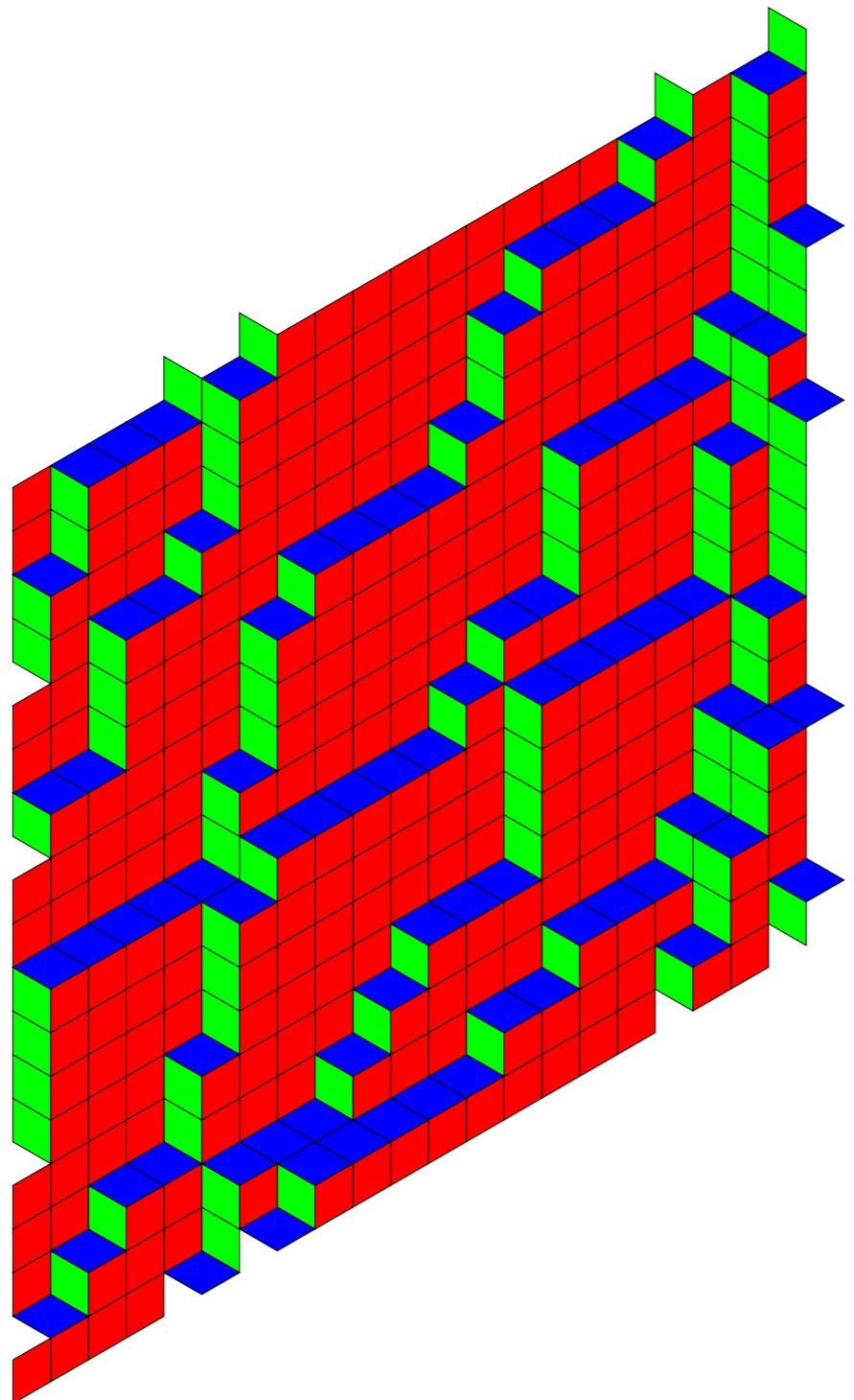
The arctic boundary is the envelope of a pencil of lines containing all boundary edges in order.

$$x \frac{(t - a_1)(t - a_2)(t - a_3)}{(t - c_1)(t - c_2)(t - c_3)} + y \frac{(t - b_1)(t - b_2)(t - b_3)}{(t - c_1)(t - c_2)(t - c_3)} = 1 \quad t \in \mathbb{R} \cup \{\infty\}$$

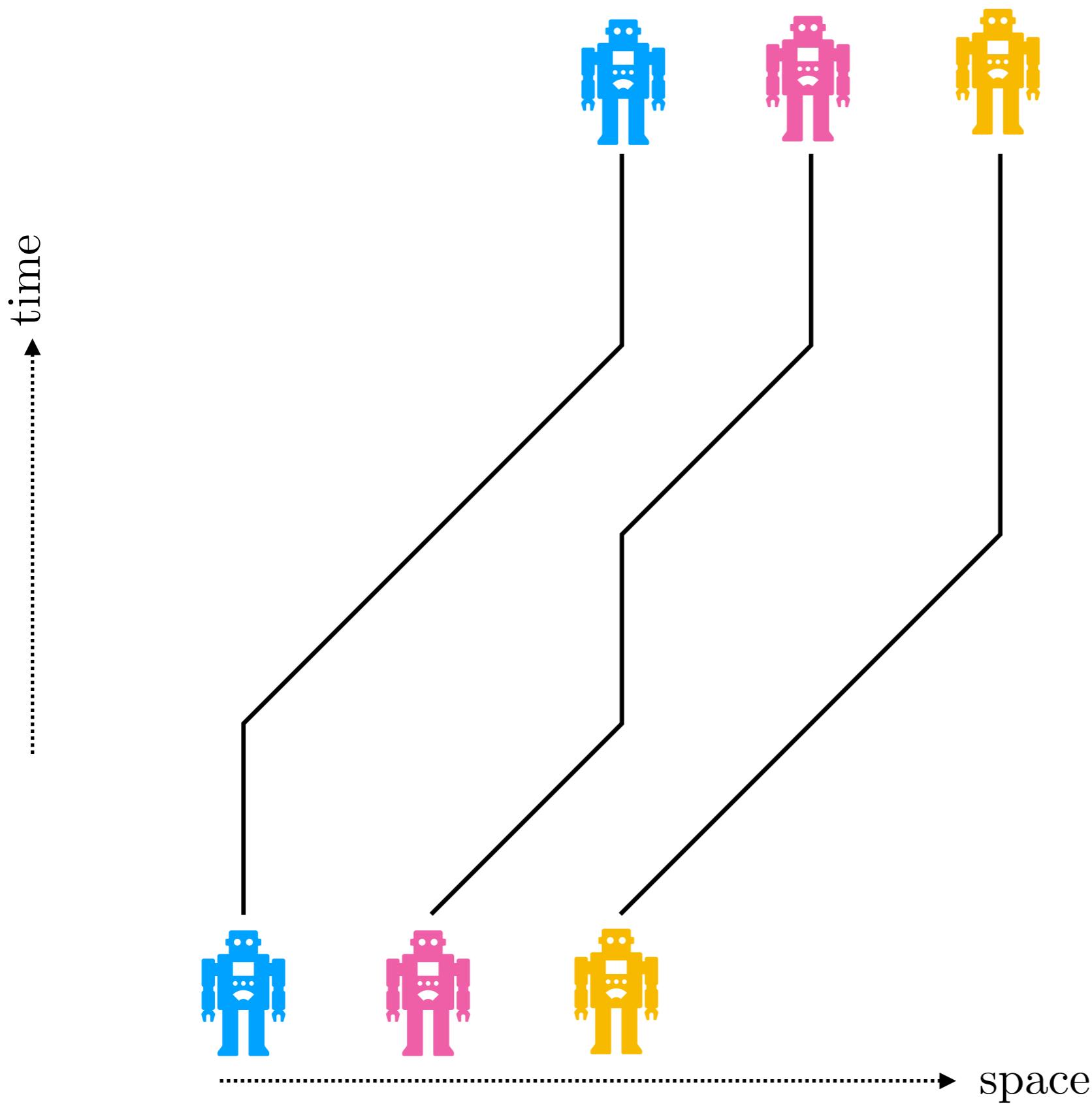
$$a_1 < b_1 < c_1 < a_2 < \cdots < c_3$$

For $t \in \mathbb{H}$, get $z = z(x, y)$ in the interior.

Lozenges and monotone nonintersecting lattice paths

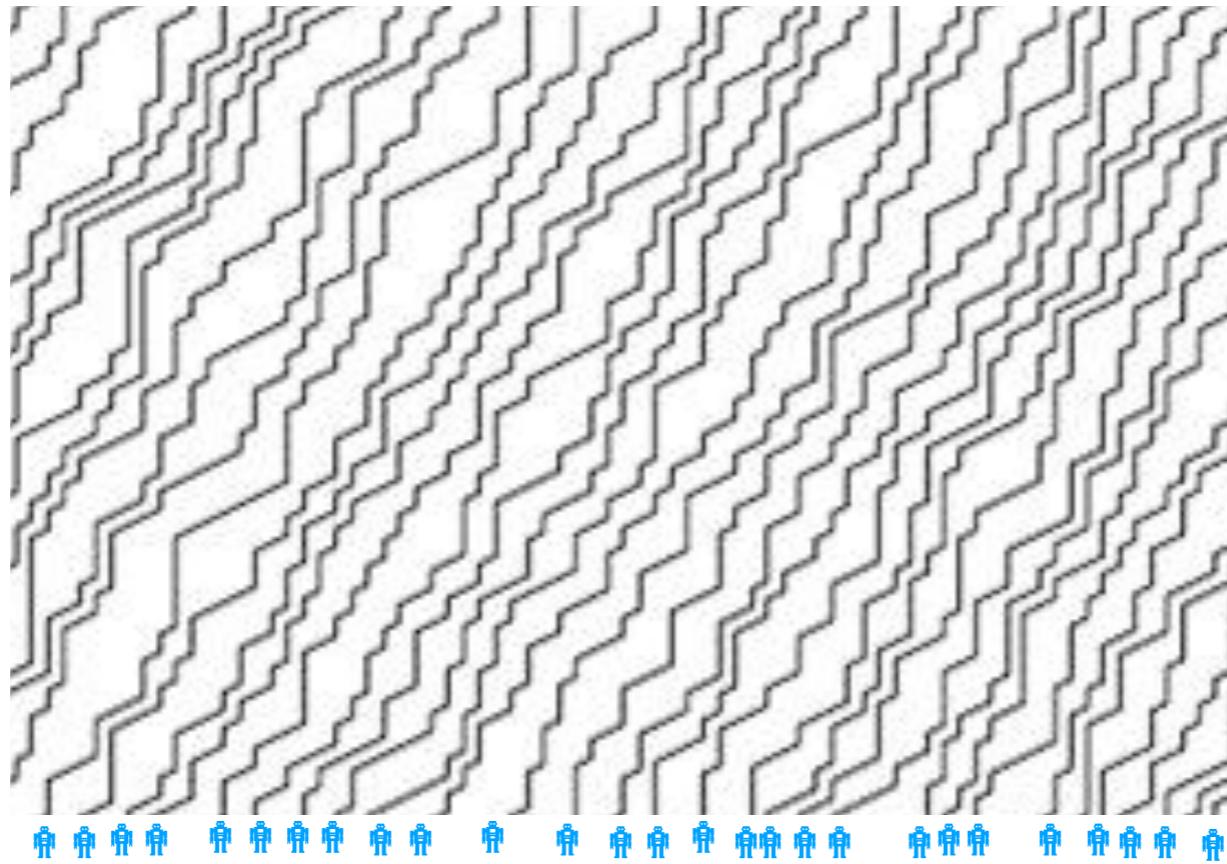


traffic flow?

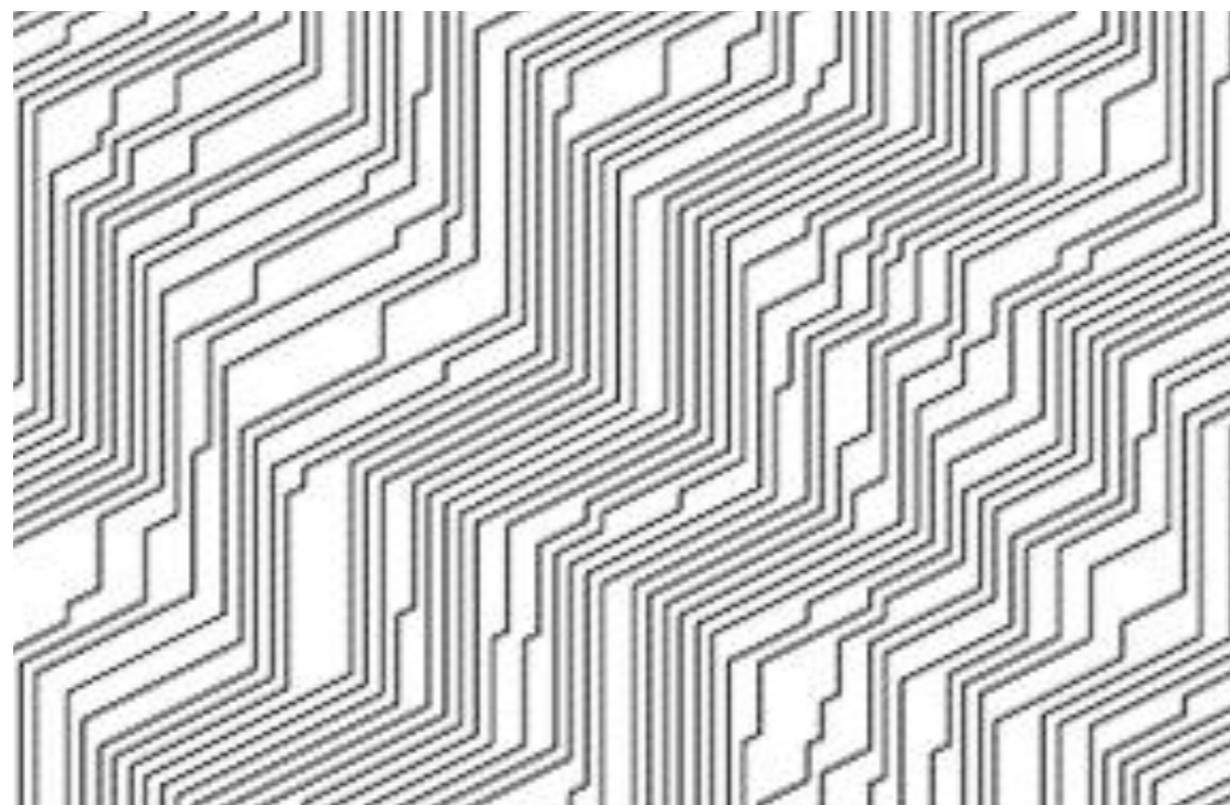


During each time unit,
speed equals 0 or 1.

if we penalize changing speed by a factor r per corner...



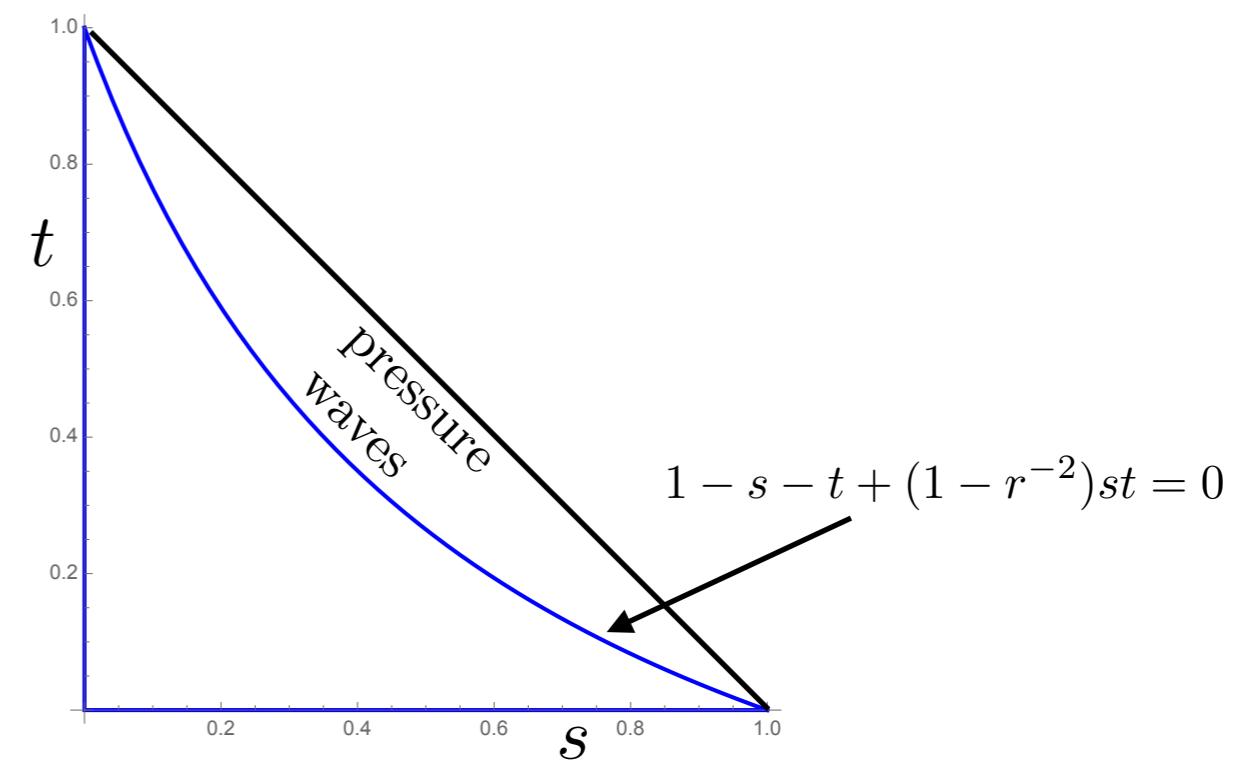
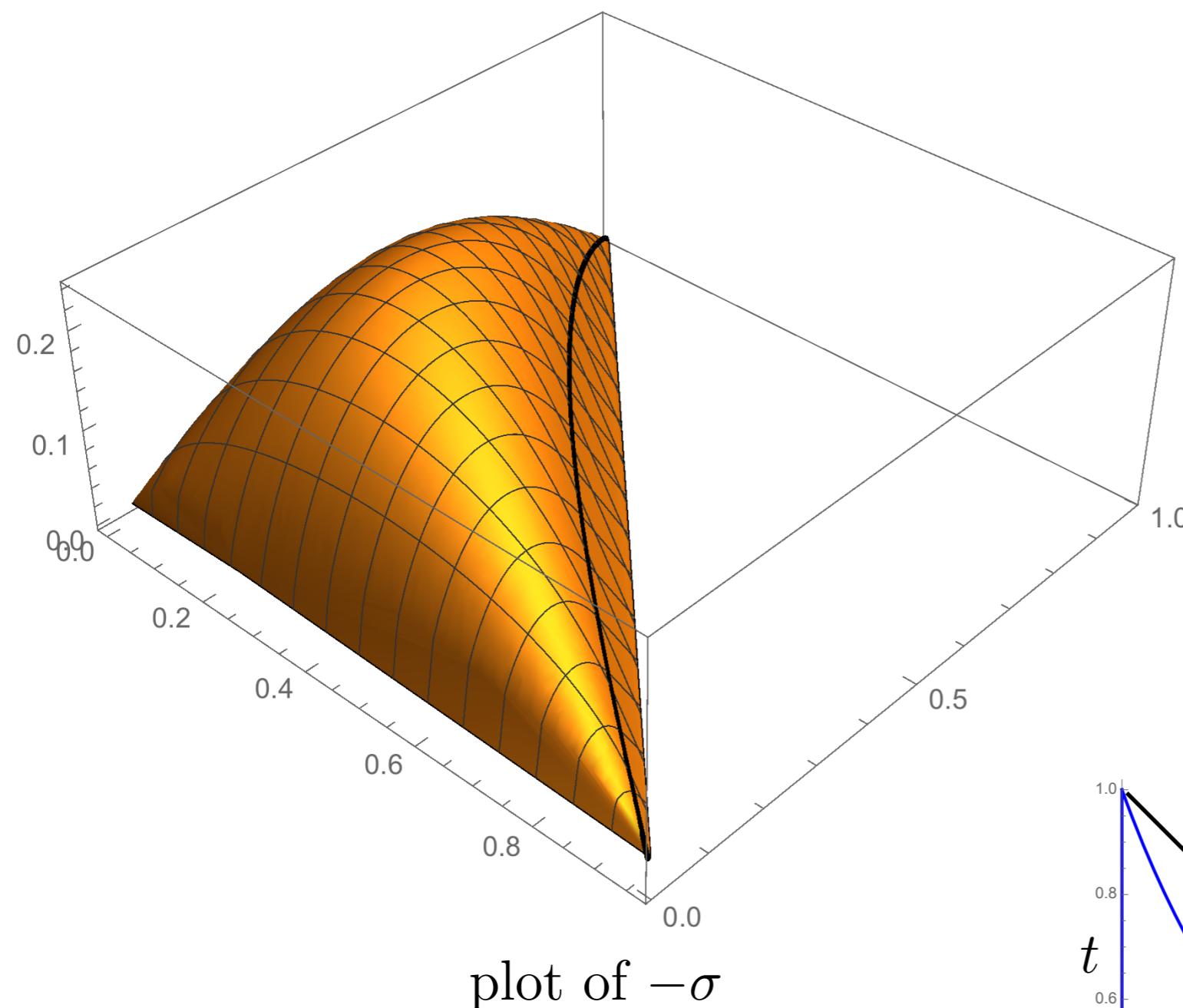
$r = 1$
(no penalty)



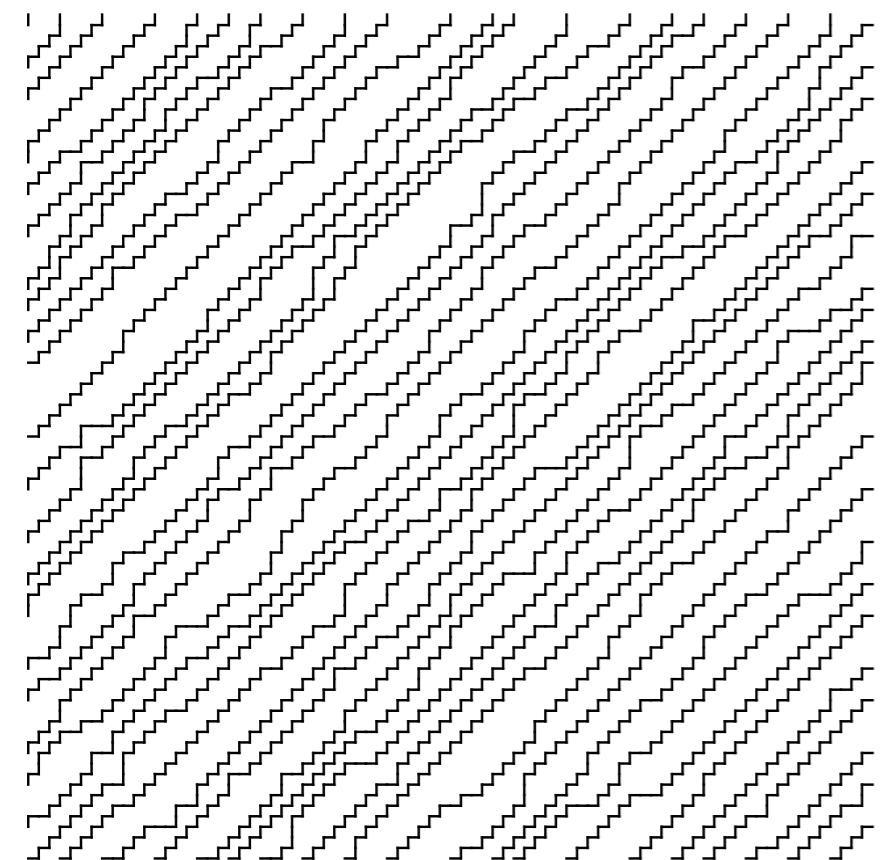
$r = .1$
(large penalty)

pressure waves

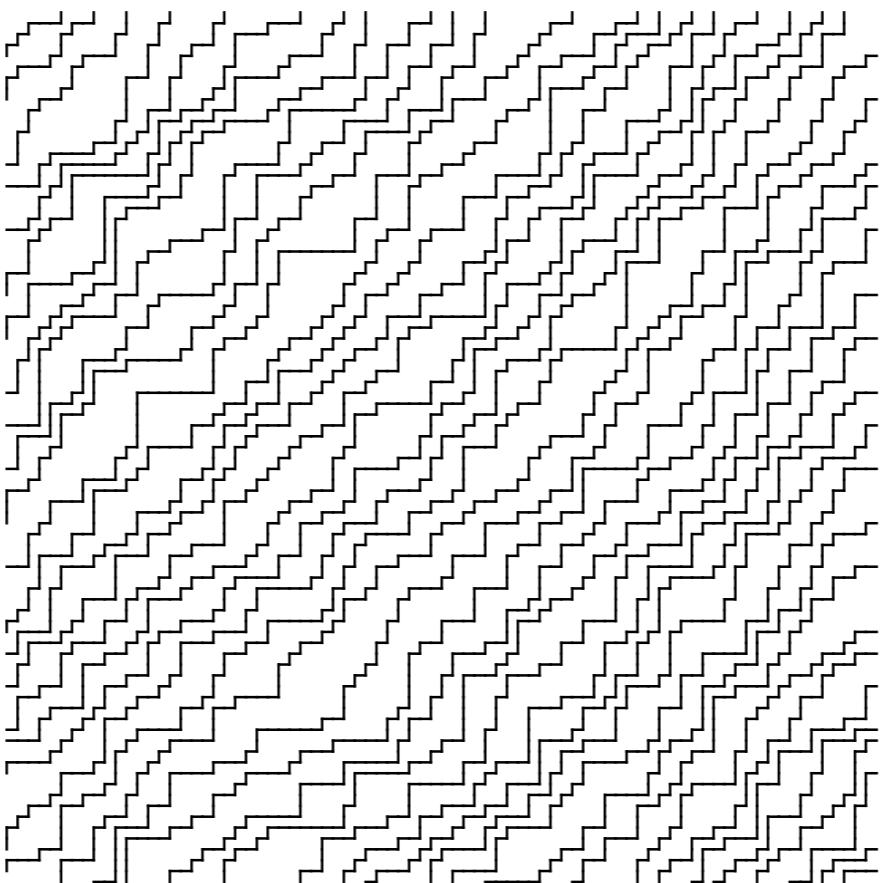
For $r < 1$, the surface tension has a different form:



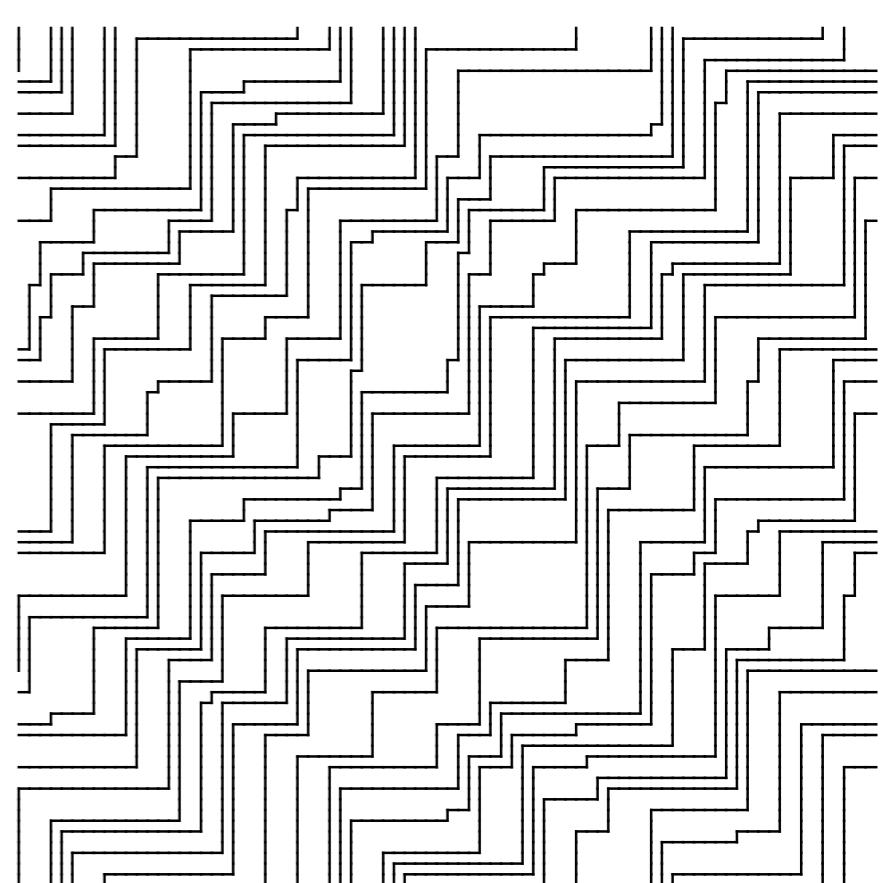
Simulations



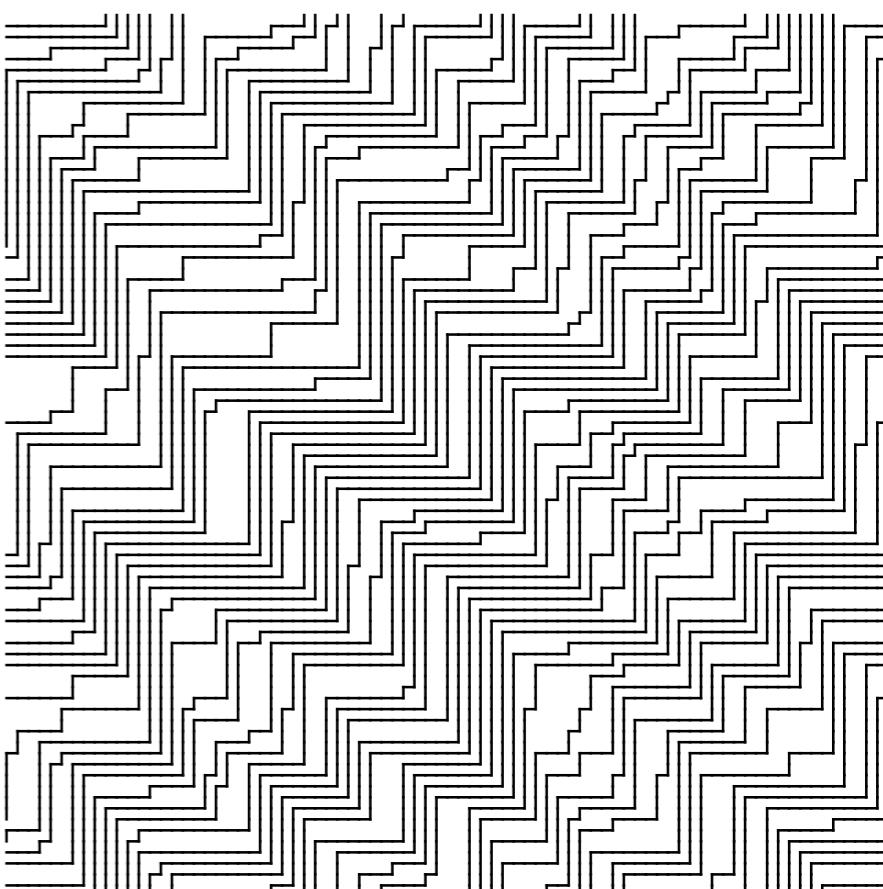
$r = 10$

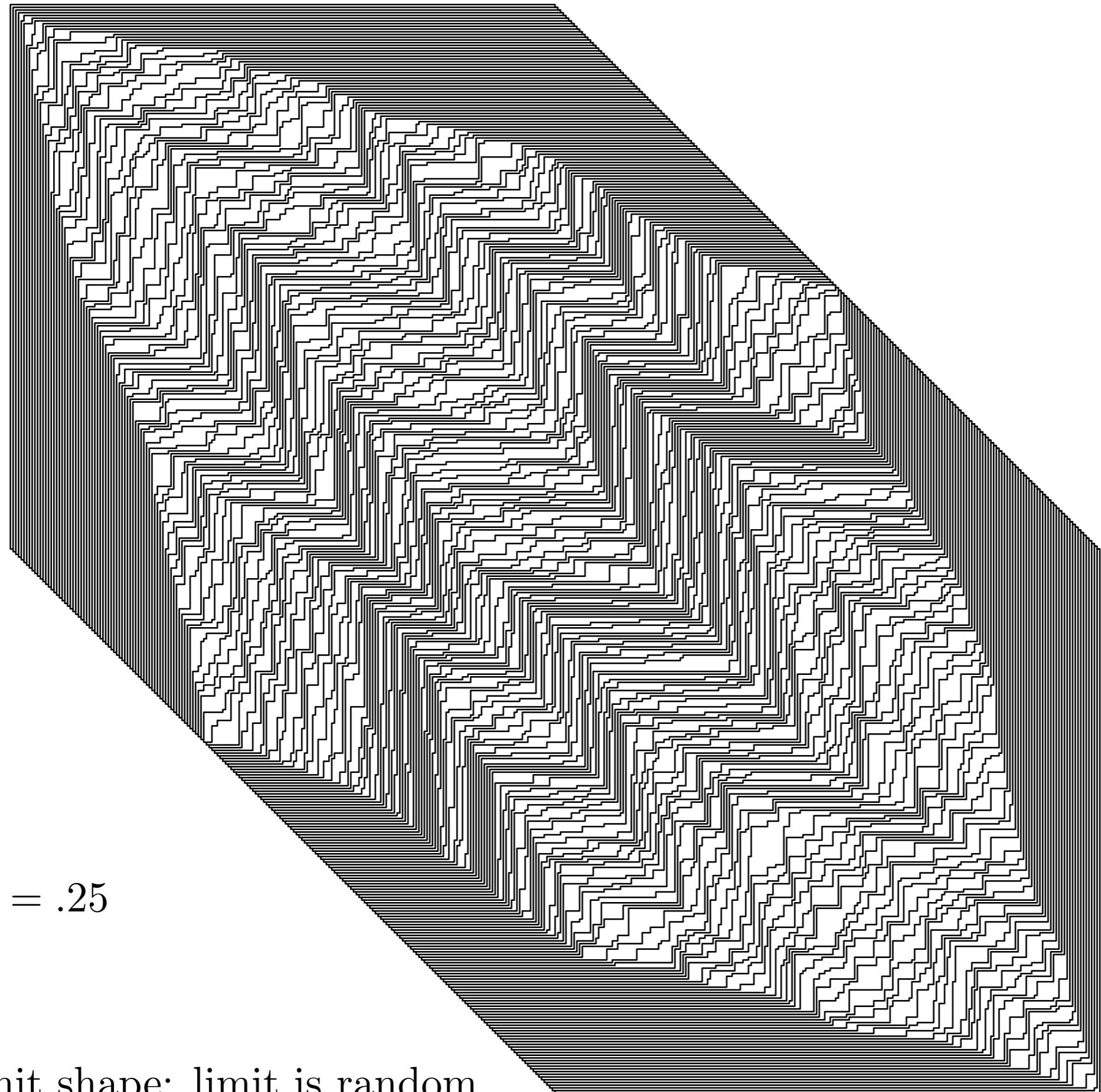


$r = 1$



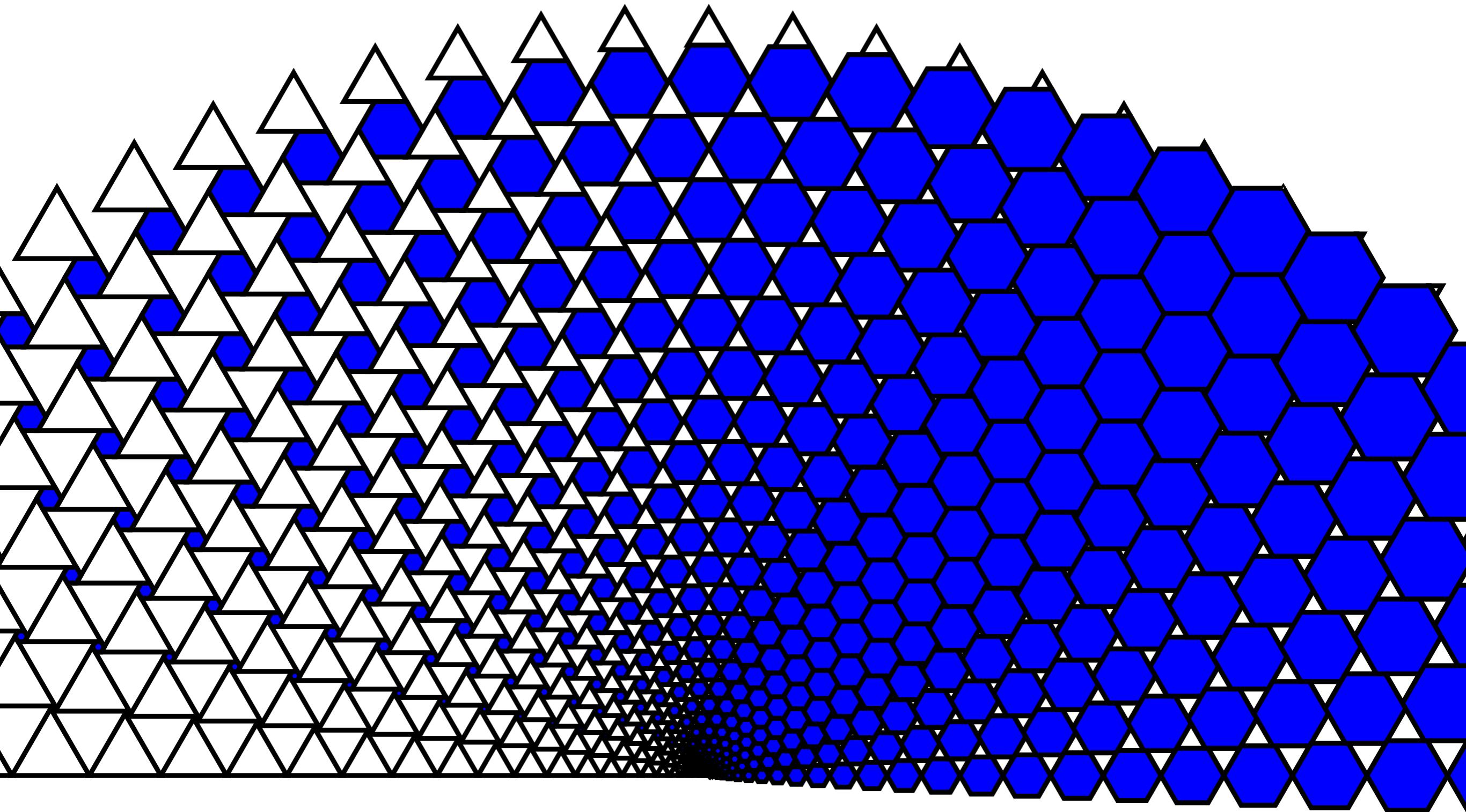
$r = .1$



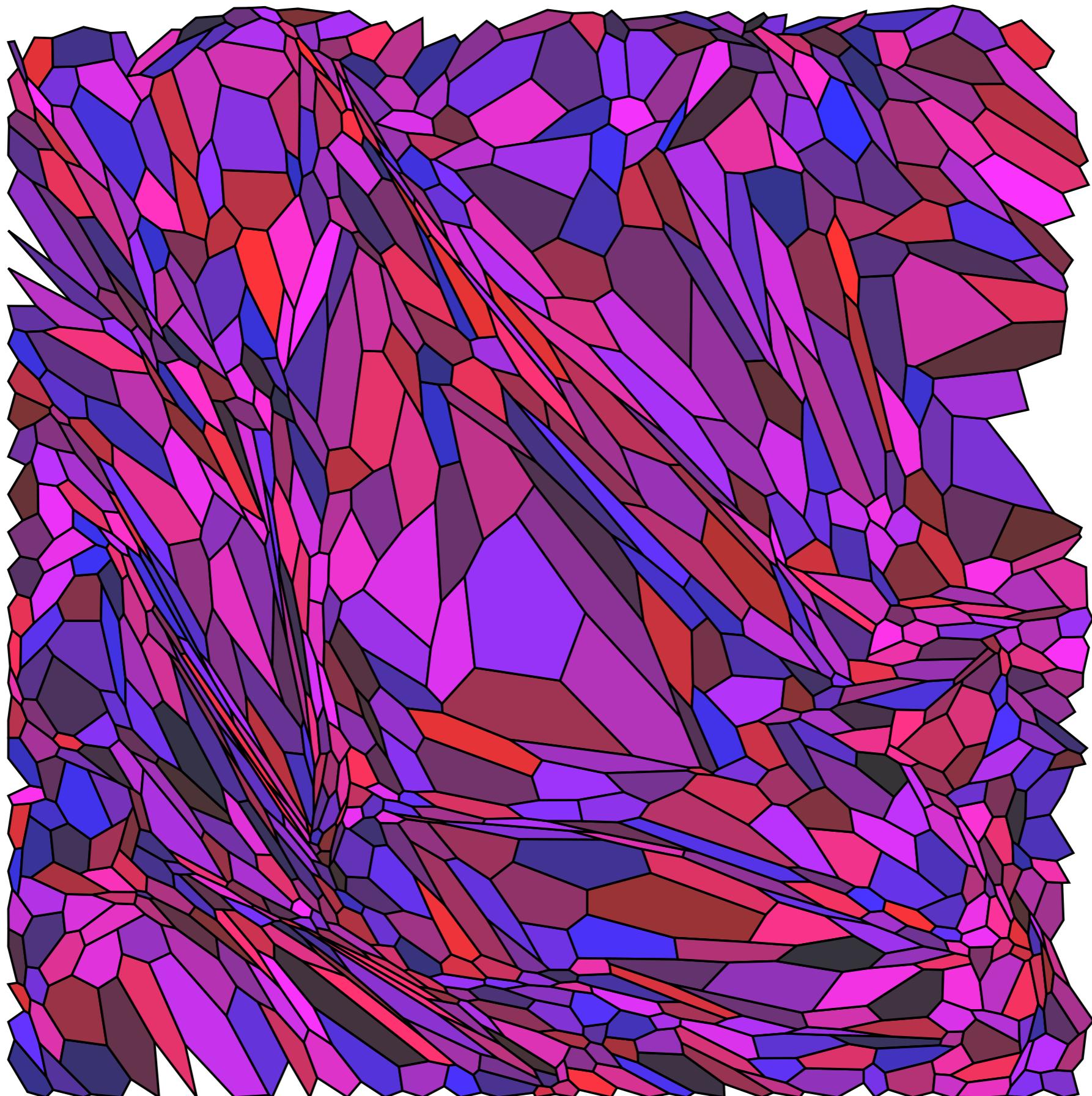


$r = .25$

No limit shape: limit is random.



thank you for your attention!



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